CHAPTER 2

# Construction of Sarmanov-Type Copulae Through Rüschendorf Method

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## 2.1 INTRODUCTION

Distributions are used to compute probability or estimate the occurrence of a random process. Therefore, coming up with distribution is crucial to modelling a particular phenomenon. In theory and practice, one needs to create a distribution incorporating specific marginals with a reasonable correlation that reflects the given data to model the involved variables' joint behaviour (Durante & Salvadori, 2010; Nelsen, 2006). But to construct one with differing marginals is a challenge. It is a problem pioneered by Eyraud (1936) and Wigner (1997) and has been an active research area even now (Lin et al., 2014). The breakthrough came when Sklar (1959) introduced the copula's notion, which revolutionized the field now known as copula theory. In essence, a copula is a distribution function that links marginals to multivariate distribution. One can study the dependence between its components (Durante & Sempi 2015) and construct a class of distributions through copula. For this reason, copula construction becomes the main interest of most mathematicians (Cuadras, 2009; Czado, 2010; Liebscher, 2008; Morillas, 2005; Nelsen et al., 2008; Wu, 2014). Some distributions generated using copula are the Bivariate Singh-Maddala and Lomax distribution, both produced by the Farlie-Gumbel-Morgenstern (FGM) (Arcede & Macalos, 2016).

Generalizations of FGM are also widely studied. For instance, Rodríguez-Lallena and Úbeda-Flores (2004) presented a family of copulas depending on two univariate functions. This family turns out to include FGM copula and its modifications (Huang & Kotz, 1999). In fact, Pathak and Vellaisamy (2016) extended the work by generalizing a subclass of such family using order statistics, including the FGM copula.

Other authors obtained a distribution of sum, product, or quotient of two random variables. We mention a few. Ly et al. (2019), Nadarajah (2008), and Nadarajah and Kotz (2006) while several others have developed other ways to generate a copula (Demarta & McNeil, 2005; Durante et al., 2008; Durante & Salvadori, 2010; Gudendorf & Segers, 2010; Joe & Kurowicka, 2011; Nelsen & Fredricks, 1997).

## 2.2 COPULA CONSTRUCTION

This section will mention the definition of copula and some essential works that led to our final method in constructing a copula. First, we will begin with the definition of copula and discuss the notion of Sarmanov distribution (Sarmanov, 1966) in the generalized context proposed by Bairamov et al. (2001). Afterward, we will compare and merge it to the approach of Rüschendorf (1985) while noting the works of Rodríguez-Lallena and Úbeda-Flores (2004).

We borrowed the excellent discussion in the paper of Lin et al. (2014). The generalized Sarmanov density copula, as introduced by Bairamov et al. (2001), possesses a simple analytical form. In a two-dimensional setting, we have

$$c(x,y) = 1 + \theta s(x,y), x,y \in [0,1] \tag{2.1}$$

where *s* is an integrable bivariate function on $[0,1]^2$ satisfying

$$\int_0^1 s(x,y)dx = \int_0^1 s(x,y)dy = 0 \tag{2.2}$$

and the copula parameter should satisfy:

$$[max\{s(x,y):(x,y) \in D^+\}]^{-1} \leq \theta \leq [max\{-s(x,y):(x,y) \in D^-\}]^{-1} \tag{2.3}$$

where

$$D^+ = \{(x,y): s(x,y) > 0\} \text{ and } D^- = \{(x,y): s(x,y) < 0\}$$

So, the generalized Sarmanov copula takes the form:

$$C(u,v) = uv + \theta \int_0^v \int_0^u s(x,y)dxdy \tag{2.4}$$

Equation (2.2) is very important to establish the boundary conditions of a copula stated in Definition 1, and Equation (2.3) is a must so that $n$-increasing property of the same is satisfied. We also note that copula parameter $\theta$ control the strength of dependence. Furthermore, it also influences the correlation coefficients Kendall's tau and Spearman's rho as functions of $\theta$. Finally, $\theta$ should depend on function *S* as stated in Equation (2.3). All these conditions should be met for *C* in Equation (2.4) to be a copula.

This method of constructing a copula has an advantage because one can decide which function *S* to choose to obtain a wider range of dependency values. Take for instance, the case of FGM copula. The correlation is restricted to the interval