CHAPTER 7

Skew-*t* Copula

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7.1 INTRODUCTION

The Copula method is introduced by Sklar (1959) to represent dependency among variables. Using the copula method, the univariate marginal can be linked to their full multivariate distribution. In other words, a copula is a multivariate distribution function for which the marginal probability distribution of each variable is uniform on the interval $[0,1]$. A multivariate distribution describes the probabilities for a group of continuous random variables. Hence, a copula is flexible in handling the dependency among the multiple variables where it generates a general structure and can link some of the variables or the properties together using a joint probability function. The probability density function (PDF) for a *d*-dimensional copula C is defined as:

$$C = \frac{\partial^d C}{\partial F_1 \ldots \partial F_d} \tag{7.1}$$

and the joint density function related to the copula density is given by:

$$f(x) = c\left(F_1(X_1), \ldots, F_d(X_d)\right) = \prod_{i=1}^{d} f_i(X_i) \tag{7.2}$$

where $c\left(F_1\left(X_1\right),\ldots,F_d\left(X_d\right)\right)$ is the *d*-dimensional copula of a joint distribution function on $\left[0,1\right]^d$ with all the marginal distributions being standard uniform distribution.

One of the limitations of working with high-dimensional data is the limited choice of multivariate distributions. Traditionally, the multivariate distributions are limited with specific distributions (Gaussian or Student's *t*), which is not appropriate for general applications such as for skewed distribution. Hansen (1994) proposed skew Student's *t* distribution with asymmetry properties of zero mean and a unit variance. The skew Student's *t* distribution is suitable to be applied in modelling rainfall which normally has the same characteristics. This chapter presents the use of the copula concept with the skew Student's *t* distribution to model the spatial dependence among variables.

7.2 SKEW-*t* DISTRIBUTION WITH COPULA CONCEPT

The skew-*t* copula model is chosen to develop a rainfall model based on the reasons mentioned in Section 7.1. This section presents the theory of univariate skew-*t* distribution, which is the basis of the univariate Student's *t* and is extended to bivariate and multivariate skew-*t* distributions. The Student's *t* distribution was introduced by W. S. Gosset in 1908, and the skew-*t* distribution is constructed based on the Student's *t* distribution.

A random variable *X* has a Student's *t* distribution with ν degrees of freedom denoted by $X \sim t\left(\nu\right)$ and the PDF is given by

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{(v+1)}{2}} \tag{7.3}$$

where $x \in (-\infty, \infty)$ and $\Gamma(.)$ defines a gamma function. The Student's t PDF can also be written in the Beta function form of

$$f(x) = \frac{1}{\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{(v+1)}{2}} \tag{7.4}$$

where $B(.)$ is the Beta function defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

with $\pi = \Gamma\left(\frac{1}{2}\right)$. The cumulative distribution function (CDF) of the Student's t is given as

$$F(x) = \frac{1}{2} + x\Gamma\left(\frac{v+1}{2}\right)\frac{\left(\frac{1}{2},\frac{(v+1)}{2}\right),\left(\frac{3}{2}-\frac{x^2}{v}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \tag{7.5}$$